HEAT TRANSFER IN FLOW OF AN INCOMPRESSIBLE VISCOUS LIQUID BETWEEN DISKS

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Inzhenerno-Fizicheskii Zhurnal, Vol. 10, No. 1, pp. 32–40, 1966

UDC 532.517.2

Using approximate equations of motion, an investigation has been made of the development of steady laminar radial flow of a viscous incompressible liquid in the gap between parallel disks. In the region of hydrodynamically stable flow the heat transfer problem is solved for a given constant heat flux at the wall.

We shall examine two plane disks, located parallel to one another at distance h. In the center of each disk there is a hole of radius r_0, through which a liquid is admitted into the gap between the disks, where it flows in a radial direction. Considering the flow between the disks to be axisymmetric, it may be described by the system of equations:

$$V_r\frac{\partial V_r}{\partial r}+V_Z\frac{\partial V_r}{\partial Z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right)+\frac{\partial^2 V_r}{\partial Z^2}-\frac{V_r}{r^2}\right],$$
$$V_r\frac{\partial V_Z}{\partial r}+V_Z\frac{\partial V_Z}{\partial Z}=-\frac{1}{\rho}\frac{\partial p}{\partial Z}+$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_Z}{\partial r}\right)+\right.$$
$$\left.+\frac{\partial^2 V_Z}{\partial Z^2}\right],\qquad \frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{\partial V_Z}{\partial Z}=0.$$

To investigate the development of longitudinal velocity, we shall use, instead of this exact system, a system of approximate equations obtained by Targ's method [1]. We simplify the initial system on the usual assumptions of boundary layer theory, and then neglect the term $V_Z(\partial V_r)/(\partial Z)$ on the left of the first equation, while in the term $V_r(\partial V_r)/(\partial r)$ we replace V_r by the average value r_0V_{r0}/r over the section (we consider that V_{r0} is constant over the section). This gives

$$\frac{r_0V_{r0}}{r}\frac{\partial V_r}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\frac{\partial^2 V_r}{\partial Z^2},\qquad(1)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial Z}=0,\qquad(2)$$

$$\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{\partial V_Z}{\partial Z}=0.\qquad(3)$$

It follows from (2) that $p = f(r)$. To determine $f(r)$ we integrate (1) with respect to Z from $-h/2$ to $h/2$ and insert the expression for the mass flow rate Q. Then

$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{Q^2}{4\pi^2r^3h^2}+\frac{2\nu}{h}\left(\frac{\partial V_r}{\partial Z}\right)_{h/2}.\qquad(4)$$

Eliminating $\frac{1}{\rho}\frac{\partial p}{\partial r}$ from (1), using (4), we obtain for the determination of v_r an equation which in dimensionless terms has the form

$$\frac{1}{R}\frac{\partial v_r}{\partial R}=k\frac{\partial^2 v_r}{\partial z^2}-\frac{1}{R^3}-k\left(\frac{\partial v_r}{\partial z}\right)_{z=1}.\qquad(5)$$

Here

$$R=r/r_0;\quad z=2Z/h;\quad v_r=V_r/V_{r0};\quad k=4\nu r_0/V_{r0}h^2.$$

The boundary conditions of the problem are

$$v_r\Big|_{z=1}=0;\quad \frac{\partial v_r}{\partial z}\Big|_{z=0}=0;\quad v_r\Big|_{R=1}=1.$$

Introducing the new independent variable $\xi = R^2 - 1$, we obtain

$$2\frac{\partial v_r}{\partial \xi}=k\frac{\partial^2 v_r}{\partial z^2}-\frac{1}{(\xi+1)^{3/2}}-k\left(\frac{\partial v_r}{\partial z}\right)_{z=1},\qquad(6)$$

$$v_r\Big|_{z=1}=0;\quad \frac{\partial v_r}{\partial z}\Big|_{z=0}=0;\quad v_r\Big|_{\xi=0}=1.\qquad(7)$$

Applying a Laplace-Carson transformation to (6) and conditions (7), we have

$$\frac{\partial^2\tilde v_r}{\partial z^2}-\frac{2S}{k}\tilde v_r=-\frac{2S}{k}\sqrt{\pi S}\exp S\,\mathrm{erf}\sqrt{S}+$$
$$+\left(\frac{\partial\tilde v_r}{\partial z}\right)_{z=1},\qquad(8)$$

$$\tilde v_r\Big|_{z=1}=0;\quad \frac{\partial\tilde v_r}{\partial z}\Big|_{z=1}=0.\qquad(9)$$

Integrating (8) with boundary conditions (9) and determining $\left(\frac{\partial\tilde v_r}{\partial z}\right)_{z=1}$, we obtain

$$\tilde v_r=\sqrt{\pi S}\exp S\,\mathrm{erf}\sqrt{S}\,\frac{\mathrm{ch}\sqrt{2S/k}\,z-\mathrm{ch}\sqrt{2S/k}}{\sqrt{k/2S}\,\mathrm{sh}\sqrt{2S/k}\,\mathrm{ch}\sqrt{2S/k}}$$

or, returning to the original,

$$v_r=\frac{3}{2}(1-z^2)\frac{1}{R}-\sum_{m=1}^{\infty}\frac{1}{\gamma_m^2}\left(1-\frac{\cos\gamma_m z}{\cos\gamma_m}\right)\times$$
$$\times\left\{2\exp\left(-\frac{k\gamma_m^2}{2}[R^2-1]\right)-\int_0^{R^2-1}\frac{\exp(-k\gamma_m^2\Theta/2)}{(R^2-\Theta)^{3/2}}\,d\Theta\right\},\qquad(10)$$

where γ_m are roots of the equation $\mathrm{tg}\gamma_m = \gamma_m$.

Let us examine the behavior of v_r as R increases. If we employ an asymptotic evaluation for the integrals on the right side of (10) [2], it may be shown that as

R increases the sum appearing in (10) diminishes as $1/R^3$. Thus, the longitudinal velocity profile as R increases approximates to the limiting profile

$$v_r = \frac{3}{2}(1-z^2)\frac{1}{R}. \qquad (11)$$

It may be seen from this expression that the dependence of the limiting profile on z is the same as in the case of a plane gap.

The presence of a limiting profile in the flow of liquid in the gap between the disks allows us to introduce the concept (analogous to flow in a plane gap or tube) of flow stabilized along the length and an entrance section. We understand stabilized flow to be flow with a longitudinal velocity profile close to the limit, and differing from it by no more than 1% [1]. Starting from this condition, the length of the entrance section, r_1, was determined. It may be seen from Fig. 2 that as r_0 increases (for given Re), the length of the entrance section increases and tends to the value for a plane gap, while as r_0 decreases it also decreases (it should be noted that a similar picture occurs in plane diffusers [1]). For $k > 20$ the length of the entrance section will be the same as for a plane gap. The dependence of L on Re is shown in Fig. 3 for several values of the parameter $\varkappa = r_0/h$. It follows from the curves presented that L increases with increase of Re, while with increase of $\varkappa$ $\partial L/\partial \mathrm{Re}$ increases.

Since the equation of motion (1) does not contain transverse velocity, the results obtained give a sufficiently accurate description only of the development of longitudinal velocity V_r. Some idea of the behavior of transverse velocity V_z may be obtained by substituting the expression for V_r into the continuity equation (3) and determining V_z from it. It is easy to show that the transverse velocity profile obtained in this way will satisfy the no-slip condition at the walls $(V_z|_{Z=\pm h/2} = 0)$. Knowing the longitudinal velocity profile, it is easy to determine the friction coefficient:

$$\lambda = -\frac{32}{\mathrm{Re}}\left(\frac{\partial v_r}{\partial z}\right)_{z=1}.$$

Using the expression for v_r, we obtain

$$\lambda = \frac{96}{\mathrm{Re}}\,\frac{1}{R} + \frac{32}{\mathrm{Re}}\sum_{m=1}^{\infty}\left\{2\exp\left[-\frac{k\gamma_m^2}{2}(R^2-1)\right] - \int_0^{R^2-1}\frac{\exp(-k\gamma_m^2\Theta/2)}{(R^2-\Theta)^{3/2}}\,d\Theta\right\}. \qquad (12)$$

The problem of heat transfer in the gap between parallel disks is solved for the case when a constant heat flux is given at the surface of the disks in the stabilized flow region (the walls of the disks are considered to be thermally insulated in the hydrodynamic entrance section). The liquid temperature at the inlet to the gap is assumed constant. The problem is solved on the assumption that the physical properties of the liquid are constant; we neglect heat flow due to heat conduction in the radial direction.

Under the assumptions indicated, the equation of heat flow, written in dimensionless variables, has the form

$$\frac{3}{16}\,\frac{1}{R}[1-z^2]\frac{\partial T}{\partial R} = \frac{\varkappa}{\mathrm{Pe}}\,\frac{\partial^2 T}{\partial z^2}, \qquad (13)$$

where

$$T = (t - t_{in})\,2\lambda/qh; \quad \mathrm{Pe} = 2hv_{r0}/a; \quad \varkappa = r_0/h.$$

The boundary conditions are

$$T\Big|_{R=R_1} = 0; \quad \frac{\partial T}{\partial z}\Big|_{z=0} = 0; \quad \frac{\partial T}{\partial z}\Big|_{z=1} = 1. \qquad (14)$$

Let us find the temperature profile in the stabilized heat transfer section T_0. If the temperature profile is fully stabilized, then

$$\frac{\partial T_0}{\partial R} = \frac{\partial T_m}{\partial R},$$

where T_m is the mean mass temperature of the liquid. From the heat balance equation, bearing in mind that T_m is zero at the inlet to the heated section, we obtain

$$T_m = \frac{4\varkappa}{\mathrm{Pe}}(R^2 - R_1^2).$$

The quantity $\Delta T = T_0 - T_m$ satisfies the equation

$$\frac{\partial^2 \Delta T}{\partial z^2} = \frac{3}{2}(1-z^2).$$

Solving this equation, and taking account of the boundary conditions (14), we obtain

$$\Delta T = \frac{3}{4}z^2\left(1-\frac{z^2}{6}\right) + C.$$

From the determination of ΔT it follows that

$$\int_0^1 \Delta T(1-z^2)\,dz = 0.$$

Substituting ΔT into this condition, we obtain $C = -39/280$. Thus,

$$T_0 = \frac{4\varkappa}{\mathrm{Pe}}(R^2 - R_1^2) + \frac{3}{4}z^2\left(1-\frac{z^2}{6}\right) - \frac{39}{280}. \qquad (15)$$

We seek a solution of (13) in the form

$$T = w + T_0, \qquad (16)$$

where ω satisfies the equation

$$\frac{3}{16}\,\frac{\mathrm{Pe}}{\varkappa}\,\frac{1}{R}(1-z^2)\frac{\partial w}{\partial R} = \frac{\partial^2 w}{\partial z^2} \qquad (17)$$

and the boundary conditions

$$w\Big|_{R=R_1} = -\left[\frac{3}{4}z^2\left(1-\frac{z^2}{6}\right) - \frac{39}{280}\right];$$

$$\frac{\partial w}{\partial z}\Big|_{z=0} = 0; \quad \frac{\partial w}{\partial z}\Big|_{z=1} = 0. \qquad (18)$$

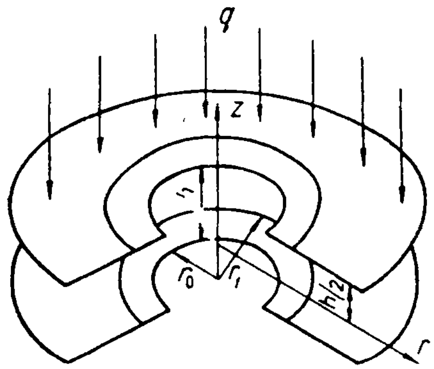

Fig. 1. Location of the coordinate axes in the gap between the disks.

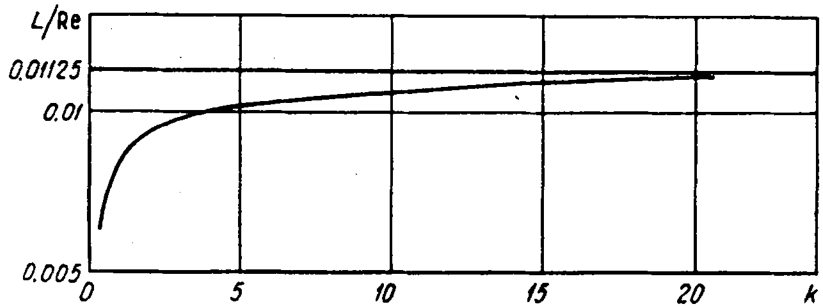

Fig. 2. Dependence of the ratio L/Re at the inlet to the gap on the parameter $k(L = r_1/2h)$.

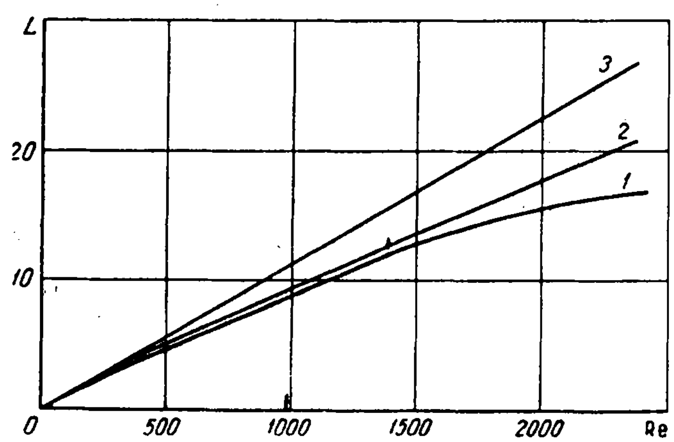

Fig. 3. Dependence of the dimensionless entrance section length L at the inlet to the gap on Re number: 1) x = 150; 2) 300; 3) plane gap.

Going over to the new variable $\xi = R^2 - R_1^2$ in (17), we obtain

$$\frac{3}{8}\frac{\mathrm{Pe}}{\varkappa}(1-z^2)\frac{\partial \omega}{\partial \xi} = \frac{\partial^2 \omega}{\partial z^2}. \qquad (19)$$

We put ω in the form

$$\omega = \Xi(\xi) Z(z). \qquad (20)$$

Z(z) satisfies the equation

$$Z'' + \lambda^2 (1-z^2) Z = 0 \qquad (21)$$

and the boundary conditions

$$\left.\frac{dZ}{dz}\right|_{z=0} = 0; \quad \left.\frac{dZ}{dz}\right|_{z=1} = 0. \qquad (22)$$

In order to determine the eigenfunctions $Z_p(z)$, we use the method described in [2].

We seek a solution of (21) in series form

$$Z_p = \sum_{n=0}^{\infty} a_n^p x_n, \qquad (23)$$

where x_n are eigenfunctions of the auxiliary equation

$$x'' + \alpha^2 x = 0, \qquad (24)$$

which satisfy the same boundary conditions, i.e.,

$$\left.\frac{dx}{dz}\right|_{z=0} = 0; \quad \left.\frac{dx}{dz}\right|_{z=1} = 0, \qquad (25)$$

where $x_n = \cos \alpha_n z$; α_n are roots of the equation $\sin \alpha_n = 0$; $n = 1, 2, \ldots$; $\alpha_n = \pi n$; $n = 1, 2, \ldots$.

If $n = 0$, and therefore $x_0 = 1$, then (23) may be conveniently written in the form

$$Z_p = \sum_{n=1}^{\infty} a_n^{(p)} x_n + a_0^{(p)}. \qquad (26)$$

An eigenfunction z_p corresponds to each eigenvalue λ_p^2.

Let us first examine the case $\lambda_p^2 = 0$. In this case the solution of (21), satisfying boundary conditions (22), takes the form z_0 = const. Since the eigenfunctions are determined to within a constant multiplier, we may consider, without loss of generality, that $z_0 = 1$. We shall now find Z_p. We multiply (21) by x_n and integrate with respect to z from 0 to 1, and find that two cases are possible:
first

$$n = 0, \quad x_0 = 1, \quad a_0^{(p)} = 3\sum_{m=1}^{\infty} a_m^{(p)} (-1)^m / \pi^2 m^2; \qquad (27)$$

second

$$n = 1, 2, \ldots, \quad x_n = \cos \pi n z,$$

$$\left\{-\frac{\pi^2 n^2}{2} + \lambda_p^2\left[\frac{1}{3} - \frac{1}{4\pi^2 n^2}\right]\right\} a_n^{(p)} -$$

$$- 2\lambda_p^2 \sum_{\substack{m=0 \\ m \neq n}}^{\infty} a_m^{(p)} (-1)^{n+m} \frac{(n^2+m^2)}{\pi^2 (n^2-m^2)^2} = 0. \qquad (28)$$

Thus, to determine the eigenvalues λ_p^2 we have a system of n equations, of which the first is (27), and the remainder have the form of (28). The number of such systems is p, which corresponds to the number of eigenvalues which we are seeking.

Substituting (27) into (28), we obtain the system

$$\left\{-\frac{\pi^2 n^2}{2} + \lambda_p^2\left[\frac{1}{3} - \frac{1}{4\pi^2 n^2}\right]\right\} a_n^p -$$

$$- 2\lambda_p^2 \sum_{m=1}^{\infty} a_m^p (-1)^{n+m} \frac{1}{\pi^2}\left[\frac{n^2+m^2}{(n^2-m^2)^2} + \right.$$

$$\left. + \frac{3}{\pi^2 n^2 m^2}\right] = 0, \qquad n = 1, 2, \ldots. \qquad (29)$$

Let us designate $f(\lambda_p, n) = -\pi^2 n^2/2 + \lambda_p^2 [1/3 - 1/4\pi^2 n^2]$ and put $a_p^{(p)} = 1$. In the case n = p (29) has the form

$$f(\lambda_p, p) - 2\lambda_p^2 \sum_{m=1}^{\infty} a_m^{(p)} (-1)^{n+m} \frac{1}{\pi^2} \times$$

$$\times \left[\frac{n^2+m^2}{(n^2-m^2)^2} + \frac{3}{\pi^2 n^2 m^2}\right] = 0, \qquad (30)$$

and in the case $n \neq p$

$$f(\lambda_p, n) a_n^{(p)} - 2\lambda_p^2 \sum_{\substack{m=1 \\ m \neq n, p}}^{\infty} a_m^{(p)} (-1)^{n+m} \frac{1}{\pi^2}\left[\frac{n^2+m^2}{(n^2-m^2)^2} + \right.$$

$$\left. + \frac{3}{\pi^2 n^2 m^2}\right] -$$

$$-2\lambda_p^2 (-1)^{n+p} \frac{1}{\pi^2}\left[\frac{n^2+p^2}{(n^2-p^2)^2} + \frac{3}{\pi^2 n^2 p^2}\right] = 0. \qquad (31)$$

In (31) the sum is of order $1/n^2$, while $f(\lambda_p, n) \sim n^2$, and therefore in the first approximation we neglect the term with the sum and obtain

$$a_n^{(p)} = 2\lambda_p^2 (-1)^{n+p} \frac{1}{\pi^2 f(\lambda_p, n)}\left[\frac{n^2+p^2}{(n^2-p^2)^2} + \right.$$

$$\left. + \frac{3}{\pi^2 n^2 p^2}\right]; \qquad n = 1, 2, \ldots; \quad n \neq p. \qquad (32)$$

In the first approximation λ_p^2 is found as a root of the equation $f(\lambda_p, p) = 0$, i.e.,

$$\lambda_p^2 = \pi^2 p^2 2 \left[\frac{1}{3} - \frac{1}{4\pi^2 p^2}\right]; \quad p = 1, 2, \ldots \qquad (33)$$

To improve on the eigenvalues λ_p^2 from (33), we substitute into (32) and find $(a_n^{(p)})_I$, which we insert in turn under the summation sign in (31) to find a more accurate value $(a_n^{(p)})_{II}$. We further substitute $(a_n^{(p)})_{II}$ under the summation sign in (30) to find new, more accurate values of λ_p^2. The process continues until the requisite accuracy is obtained.

After the eigenvalues λ_p^2 are found, the eigenfunctions Z_p are found in series form (26).

Let us find the functions $\Xi_p(\xi)$, which satisfy the equation

$$\frac{d\Xi_p}{d\xi} = -\lambda_p^2 \frac{8}{3} \frac{\varkappa}{\mathrm{Pe}} \Xi_p; \quad \Xi_p = \exp\left(-\frac{8}{3}\frac{\varkappa}{\mathrm{Pe}}\lambda_p^2 \xi\right)$$

to within a constant multiplier.

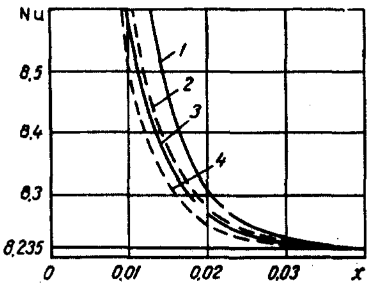

Fig. 4. Dependence of Nu = $(4)/(T_s - T_m)$ on $x = (r - r_1)/(2h)(\mathrm{Pe})^{-1}$: 1) plane gap; 2) Pe = 500, $\varkappa$ = 300; 3) Pe = 500, $\varkappa$ = 150; 4) Pe = 1000, $\varkappa$ = 150.

Thus,

$$w = \sum_{p=0}^{\infty} \Xi_p Z_p = \sum_{p=1}^{\infty} C_p \exp\left(-\frac{8}{3}\frac{\varkappa}{\mathrm{Pe}}\lambda_p^2 \xi\right) \times \left[\sum_{n=1}^{\infty} a_n^p \cos \pi n z + a_0\right] + C_0, \tag{34}$$

where C_p are constants which are found from the condition at the inlet to the heated section:

$$w\Big|_{\xi=0} = -\left[\frac{3}{4} z^2 \left(1 - \frac{z^2}{6}\right) - \frac{39}{280}\right] = \varphi(z); \quad C_0 = 0; \tag{35}$$

$$C_p = \sum_{n=1}^{\infty} a_n^{(p)} (-1)^{n+1} \frac{2}{\pi^2 n^2}\left(-\frac{17}{35} + \frac{3}{\pi^2 n^2} + \frac{45}{\pi^4 n^4}\right) \times \left\{\sum_{n=1}^{\infty} (a_n^{(p)})^2 \left(\frac{1}{3} - \frac{1}{4\pi^2 n^2}\right) - 4 \sum_{n,m=1}^{\infty} (-1)^{n+m} \frac{n^2 + m^2}{\pi^2 (n^2 - m^2)^2} a_n^{(p)} a_m^{(p)} - \frac{2}{3} a_0^2\right\}^{-1}. \tag{36}$$

The final expression for the liquid temperature has the form

$$T = 4\frac{\varkappa}{\mathrm{Pe}}\left[R^2 - R_1^2\right] + \frac{3}{4} z^2 - \frac{1}{8} z^4 - \frac{39}{280} + \sum_{p=1}^{\infty} C_p \exp\left(-\frac{8}{3}\frac{\varkappa}{\mathrm{Pe}}\lambda_p^2 [R^2 - R_1^2]\right)\left(\sum_{n=1}^{\infty} a_n^p \cos \pi n z + a_0\right).$$

Let us determine the law of variation of local Nu number with radius

$$\mathrm{Nu} = 4\,(T|_{z=1} - T_m); \quad T_m = \int_0^1 T V_r dz \Big/ \int_0^1 V_r dz.$$

Using the expression for T, we obtain

$$\mathrm{Nu} = 4\left\{17/35 + \sum_{p=1}^{\infty} C_p \exp\left(-\frac{8}{3}\frac{\varkappa}{\mathrm{Pe}}\lambda_p^2 [R^2 - R_1^2]\right) \times \left(\sum_{n=1}^{\infty} a_n^{(p)} (-1)^n + a_0\right)\right\}^{-1}.$$

As R increases, Nu → 8.235. Thus, the stabilized value of Nu number for the flow of an incompressible liquid in the gap between the disks, under constant heat flux, coincides with the stabilized value of Nu for a plane gap with q = const.

The values found for the first eigenvalues were: λ_1^2 = 15.18289 (fourth approximation) and λ_2^2 = = 65.35358 (seventh approximation).

The corresponding coefficients $a_n^{(p)}$ and C_p are $a_0^{(1)} = -0.293733$; $a_0^{(2)} = 0.214582$; $a_1^{(1)} = +1$; $a_1^{(2)} = -0.496348$; $a_2^{(1)} = 0.12981088$; $a_2^{(2)} = 1$; $a_3^{(1)} = -0.0092\text{-}56616$; $a_3^{(2)} = 0.35723$ $a_4^{(1)} = 0.0028332$; $a_4^{(2)} = -0.0078\text{-}2605$; $a_5^{(2)} = 0.005513$; $C_1 = 0.20795$; $C_2 = -0.0323171$.

Figure 4 shows the variation of Nu along the radius of the gap for various values of Pe and $\varkappa$. It may be seen from the curves presented that with increase of $\varkappa$ the heat transfer in the gap between the disks approximates to that in a plane gap.

It follows from the results obtained that the hydrodynamics and heat transfer of the flow in the gap between the disks has much in common with the plane gap case. As r_0 increases, the results obtained go over to the analogous results for a plane gap. This is easily confirmed by making the limiting transition when $r_0 \to \infty$, $r \to \infty$, $r/r_0 = 1$ at a fixed value of $r - r_0$.

NOTATION

r, Z—cooordinates in radial and axial directions; r_0—radius of central hole; r_1—boundary of hydrodynamic entrance section; h—gap width; V_r, V_z—velocity of liquid in radial and axial directions; V_{r0} velocity of liquid at inlet to gap; p—pressure; t—temperature; t_{in}—liquid temperature at gap inlet; q—heat flux; ρ—density; ν—kinematic viscosity; a—thermal diffusivity; S—Laplace-Carson transform parameter; L—dimensionless length of entrance section, L = = $r_1/2h$.

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16 March 1965